

Some Sharpening and Generalizations of a result of T. J. Rivlin*

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Abstract. Let $p(z) = a_0 + a_1z + a_2z^2 + a_3z^3 + \cdots + a_nz^n$ be a polynomial of degree n . Rivlin [12] proved that if $p(z) \neq 0$ in the unit disk, then for $0 < r \leq 1$, $\max_{|z|=r} |p(z)| \geq \left(\frac{r+1}{2}\right)^n \max_{|z|=1} |p(z)|$.

In this paper, we prove a sharpening and generalization of this result, and show by means of examples that for some polynomials our result can significantly improve the bound obtained by the Rivlin's Theorem.

Key Words: Inequalities, Polynomials, Zeros.

AMS Subject Classifications: 15A18; 30C10; 30C15; 30A10

Chinese Library Classifications: O175.27

1 Introduction

Let $p(z) = \sum_{j=0}^n a_j z^j$ be a polynomial of degree n , $M(p, r) := \max_{|z|=r} |p(z)|$, $r > 0$,

$\|p\| := \max_{|z|=1} |p(z)|$, and $D(0, K) := \{z : |z| < K\}$, $K > 0$. Then it is well known that

$$M(p', 1) \leq n\|p\|, \quad (1.1)$$

and

$$M(p, R) \leq R^n \|p\|, \quad R \geq 1. \quad (1.2)$$

The above inequalities are known as Bernstein inequalities, and have been the starting point of a considerable literature in approximation theory. Several papers and research monographs have been written on this subject (see, for example Govil and Mohapatra [3], Milovanović, Mitrinović and Rassias [6], Rahman [9], Nwaeze [7], and Rahman

*This is a preprint of a paper whose final and definite form is published in "Analysis in Theory and Applications."

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and Schmeisser [10,11]).

For polynomials of degree n not vanishing in the interior of the unit circle, the above inequalities have been replaced by:

$$M(p', 1) \leq \frac{n}{2} \|p\|,$$

and

$$M(p, R) \leq \left(\frac{R^n + 1}{2} \right) \|p\|, \quad R \geq 1.$$

If one applies Inequality (1.2) to the polynomial $P(z) = z^n p(1/z)$ and use maximum modulus principle, one easily gets

Theorem 1.1. Let $p(z) = \sum_{j=0}^n a_j z^j$ be a polynomial of degree n . Then for $0 < r \leq 1$,

$$M(p, r) \geq r^n \|p\|. \quad (1.3)$$

Equality holds for $p(z) = \alpha z^n$, α being a complex number.

The above result is due to Varga [13] who attributes it to E. H. Zarantonello.

It was shown by Govil, Qazi and Rahman [4] that the inequalities (1.1), (1.2) and (1.3) are all equivalent in the sense that any of these inequalities can be derived from the other.

The analogue of Inequality (1.3) for polynomials not vanishing in the interior of a unit circle was proved in 1960 by Rivlin [12], who in fact proved

Theorem 1.2 (Rivlin [12]). Let $p(z) = \sum_{j=0}^n a_j z^j \neq 0$ in $D(0,1)$. Then for $0 < r \leq 1$,

$$M(p, r) \geq \left(\frac{r+1}{2} \right)^n \|p\|.$$

The inequality is best possible and equality holds for $p(z) = \left(\frac{\alpha + \beta z}{2} \right)^n$, where $|\alpha| = |\beta| = 1$.

Govil [1] generalized Theorem 1.2 by proving

Theorem 1.3. Let $p(z) = \sum_{j=0}^n a_j z^j \neq 0$ in $D(0,1)$. Then for $0 < r \leq \rho \leq 1$,

$$M(p, r) \geq \left(\frac{1+r}{1+\rho} \right)^n M(p, \rho). \quad (1.4)$$

The result is best possible and equality holds for the polynomial $p(z) = \left(\frac{1+z}{1+\rho} \right)^n$.

There are many extensions of Inequality (1.4) (See, for example Govil, Qazi and Rahman [4], Govil and Qazi [5], and Qazi [8]). Also, for some more results in this direction, see Zireh et al. [14–16].

In this paper, we present some further extensions and sharpening of Rivlin's result, Theorem 1.2.

2 Main Results

Our first result is the following which, besides generalizing and sharpening several results in this direction, generalizes and sharpens Theorem 1.2 due to Rivlin [12].

Theorem 2.1. Let $p(z) = a_0 + \sum_{j=\mu}^n a_j z^j$, $1 \leq \mu < n$. If $p(z) \neq 0$ in $|z| < 1$, then for $0 < r < 1$,

$$M(p, r) \geq \frac{(1+r)^{n/\mu}}{(1+r^\mu)^{n/\mu} + \mu 2^{n/\mu} - \mu(1+r)^{n/\mu}} \left[M(p, 1) + n \min_{|z|=1} |p(z)| \ln \left(\frac{2}{1+r} \right) \right].$$

The above inequality becomes equality for the polynomial $p(z) = (1+z)^n$.

If $p(z)$ is a polynomial of degree n having no zeros in $|z| < K$, $K > 0$, then the polynomial $P(z) = p(Kz) \neq 0$ for $|z| < 1$. Further, if $0 < r < K$, then $0 < r/K < 1$, and applying Theorem 2.1 to $P(z)$, we get

$$M(P, r/K) \geq \frac{(1+r/K)^{n/\mu}}{(1+(r/K)^\mu)^{n/\mu} + \mu 2^{n/\mu} - \mu(1+r/K)^{n/\mu}} \left[M(P, 1) + n \min_{|z|=1} |P(z)| \ln \left(\frac{2}{1+r/K} \right) \right],$$

which yields

$$M(p, r) \geq \frac{K^{-n/\mu} (r+K)^{n/\mu}}{K^{-n} (r^\mu + K^\mu)^{n/\mu} + \mu 2^{n/\mu} - \mu K^{-n/\mu} (r+K)^{n/\mu}} \left[M(p, K) + nm \ln \left(\frac{2K}{r+K} \right) \right],$$

where $m = \min_{|z|=K} |p(z)|$.

This, in fact, leads to the following more general result.

Theorem 2.2. Let $p(z) = a_0 + \sum_{j=\mu}^n a_j z^j$, $1 \leq \mu < n$. If $p(z) \neq 0$ in $|z| < K$, $K > 0$, then for $0 < r < K$,

$$M(p, r) \geq \frac{K^{-n/\mu} (r+K)^{n/\mu}}{K^{-n} (r^\mu + K^\mu)^{n/\mu} + \mu 2^{n/\mu} - \mu K^{-n/\mu} (r+K)^{n/\mu}} \left[M(p, K) + nm \ln \left(\frac{2K}{r+K} \right) \right],$$

where $m = \min_{|z|=K} |p(z)|$. Again, the equality holds for the polynomial $p(z) = (1+z)^n$.

As a generalization and sharpening of Theorem 1.3, we will be proving

Theorem 2.3. Let $p(z) = \sum_{j=0}^n a_j z^j$. If $p(z) \neq 0$ in $|z| < K$, $K \geq 1$, then for $0 < r < R \leq 1$,

$$M(p, r) \geq \frac{(1+r)^n}{(1+r)^n + (R+K)^n - (r+K)^n} \left[M(p, R) + nm \ln \left(\frac{R+K}{r+K} \right) \right],$$

where $m = \min_{|z|=K} |p(z)|$.

On taking $K=1$, the above theorem reduces to

Corollary 2.1. Let $p(z) = \sum_{j=0}^n a_j z^j$. If $p(z) \neq 0$ in $|z| < 1$, then for $0 < r < R \leq 1$,

$$M(p, r) \geq \left(\frac{1+r}{1+R} \right)^n \left[M(p, R) + nm \ln \left(\frac{1+R}{1+r} \right) \right],$$

where $m = \min_{|z|=1} |p(z)|$.

Clearly, the above corollary sharpens Theorem 1.3 due to Govil [1].

If we take $R=1$, in Theorem 2.3, we get

Corollary 2.2. Let $p(z) = \sum_{j=0}^n a_j z^j$. If $p(z) \neq 0$ in $|z| < K$, $K \geq 1$, then for $0 < r < 1$,

$$M(p, r) \geq \frac{(1+r)^n}{(1+r)^n + (1+K)^n - (r+K)^n} \left[M(p, 1) + n \min_{|z|=K} |p(z)| \ln \left(\frac{1+K}{r+K} \right) \right].$$

Setting $K=1$ in Corollary 2.2 gives

Corollary 2.3. Let $p(z) = \sum_{j=0}^n a_j z^j$. If $p(z) \neq 0$ in $|z| < 1$, then for $0 < r < 1$,

$$M(p, r) \geq \left(\frac{1+r}{2} \right)^n \left[M(p, 1) + n \min_{|z|=1} |p(z)| \ln \left(\frac{2}{1+r} \right) \right].$$

The above corollary clearly sharpens Theorem 1.2 due to Rivlin [12], and excepting the case when $\min_{|z|=1} |p(z)| = 0$, the Corollary 2.3 always gives a bound that is sharper than the bound obtainable from Theorem 1.2.

3 Lemmas

For the proofs of Theorems 2.1 and 2.3, we will need the following lemmas.

In this direction, our first lemma is a result due to Govil [2, Corollary 1].

Lemma 3.1. *Let $p(z)$ be a polynomial of degree n having no zeros in $|z| < K$, $K \geq 1$, then*

$$\max_{|z|=1} |p'(z)| \leq \frac{n}{1+K} \left[\max_{|z|=1} |p(z)| - \min_{|z|=K} |p(z)| \right].$$

Lemma 3.2 (Qazi [8]). *Let $p(z) = a_0 + \sum_{j=\mu}^n a_j z^j$, $1 \leq \mu < n$. If $p(z) \neq 0$ for $|z| < 1$, then for $0 < r < R \leq 1$,*

$$M(p, r) \geq \left(\frac{1+r^\mu}{1+R^\mu} \right)^{n/\mu} M(p, R);$$

more precisely,

$$M(p, r) \geq \exp \left(-n \int_r^R \frac{t^\mu + (\mu/n) |a_\mu/a_0| t^{\mu-1}}{t^{\mu+1} + (\mu/n) |a_\mu/a_0| (t^\mu + t) + 1} dt \right) M(p, R).$$

4 Proofs

Proof of Theorem 2.1. Let $0 < r < 1$, and $\theta \in [0, 2\pi)$. Then we have:

$$|p(e^{i\theta}) - p(re^{i\theta})| = \left| \int_r^1 e^{i\theta} p'(te^{i\theta}) dt \right|,$$

which implies

$$|p(e^{i\theta})| \leq |p(re^{i\theta})| + \left| \int_r^1 e^{i\theta} p'(te^{i\theta}) dt \right|. \quad (4.1)$$

If $p(z) \neq 0$ in $|z| < 1$, then $p(tz) \neq 0$ in $|z| < 1/t$. Further, if $0 < t \leq 1$, then $1/t \geq 1$ and hence by Lemma 3.1 we get

$$t|p'(tz)| \leq \frac{nt}{1+t} \left[M(p, t) - \min_{|z|=1} |p(z)| \right]$$

which is equivalent to

$$|p'(tz)| \leq \frac{n}{1+t} \left[M(p, t) - \min_{|z|=1} |p(z)| \right]. \quad (4.2)$$

Combining (4.1) and (4.2) yield

$$|p(e^{i\theta})| \leq |p(re^{i\theta})| + \int_r^1 \frac{n}{1+t} M(p, t) dt - n \min_{|z|=1} |p(z)| \int_r^1 \frac{1}{1+t} dt.$$

which clearly gives,

$$M(p,1) \leq M(p,r) + \int_r^1 \frac{n}{1+t} M(p,t) dt - n \min_{|z|=1} |p(z)| \int_r^1 \frac{1}{1+t} dt.$$

On applying Lemma 3.2 and noting that $0 < r < t < 1$, we obtain

$$\begin{aligned} M(p,1) &\leq M(p,r) + \int_r^1 \frac{n}{1+t} \left(\frac{1+t^\mu}{1+r^\mu} \right)^{n/\mu} M(p,r) dt - n \min_{|z|=1} |p(z)| \int_r^1 \frac{1}{1+t} dt \\ &\leq M(p,r) + \int_r^1 \frac{n}{1+t} \left(\frac{1+t}{1+r^\mu} \right)^{n/\mu} M(p,r) dt - n \min_{|z|=1} |p(z)| \int_r^1 \frac{1}{1+t} dt \\ &= M(p,r) + \frac{nM(p,r)}{(1+r^\mu)^{n/\mu}} \int_r^1 \frac{(1+t)^{n/\mu}}{1+t} dt - n \min_{|z|=1} |p(z)| \int_r^1 \frac{1}{1+t} dt \\ &= M(p,r) + \frac{nM(p,r)}{(1+r^\mu)^{n/\mu}} \left[2^{n/\mu} - (1+r)^{n/\mu} \right] \frac{\mu}{n} - n \min_{|z|=1} |p(z)| \int_r^1 \frac{1}{1+t} dt \\ &= M(p,r) + \frac{\mu M(p,r)}{(1+r^\mu)^{n/\mu}} \left[2^{n/\mu} - (1+r)^{n/\mu} \right] - n \min_{|z|=1} |p(z)| \ln \left(\frac{2}{1+r} \right). \end{aligned}$$

Thus we get

$$M(p,r) \left[1 + \frac{\mu 2^{n/\mu}}{(1+r^\mu)^{n/\mu}} - \frac{\mu(1+r)^{n/\mu}}{(1+r^\mu)^{n/\mu}} \right] \geq M(p,1) + n \min_{|z|=1} |p(z)| \ln \left(\frac{2}{1+r} \right)$$

which implies

$$M(p,r) \left[\frac{(1+r^\mu)^{n/\mu} + \mu 2^{n/\mu} - \mu(1+r)^{n/\mu}}{(1+r^\mu)^{n/\mu}} \right] \geq M(p,1) + n \min_{|z|=1} |p(z)| \ln \left(\frac{2}{1+r} \right).$$

The above is clearly equivalent to

$$M(p,r) \geq \frac{(1+r^\mu)^{n/\mu}}{(1+r^\mu)^{n/\mu} + \mu 2^{n/\mu} - \mu(1+r)^{n/\mu}} \left[M(p,1) + n \min_{|z|=1} |p(z)| \ln \left(\frac{2}{1+r} \right) \right],$$

and this completes the proof of the theorem. \square

Proof of Theorem 2.3. As in the proof of Theorem 2.1, we obtain similarly that

$$|p(Re^{i\theta})| \leq |p(re^{i\theta})| + \left| \int_r^R e^{i\theta} p'(te^{i\theta}) dt \right|. \quad (4.3)$$

Now if $p(z) \neq 0$ in $|z| < K$, $K \geq 1$, then $p(tz) \neq 0$ in $|z| < K/t$. Further, if $0 < t \leq 1$, then $1/t \geq 1$ and $K/t \geq 1$.

By Lemma 3.1, we get

$$|p'(tz)| \leq \frac{n}{K+t} \left[M(p,t) - \min_{|z|=K} |p(z)| \right]. \quad (4.4)$$

Using relations (4.3) and (4.4), we get

$$|p(Re^{i\theta})| \leq |p(re^{i\theta})| + \int_r^R \frac{n}{K+t} M(p,t) dt - n \min_{|z|=K} |p(z)| \int_r^R \frac{1}{K+t} dt,$$

which implies

$$M(p,R) \leq M(p,r) + \int_r^R \frac{n}{K+t} M(p,t) dt - n \min_{|z|=K} |p(z)| \int_r^R \frac{1}{K+t} dt.$$

Now using Lemma 3.2, we obtain

$$\begin{aligned} M(p,R) &\leq M(p,r) + \int_r^R \frac{n}{K+t} \left(\frac{1+t}{1+r} \right)^n M(p,r) dt - n \min_{|z|=K} |p(z)| \int_r^R \frac{1}{K+t} dt \\ &= M(p,r) + \frac{nM(p,r)}{(1+r)^n} \int_r^R \frac{(1+t)^n}{K+t} dt - n \min_{|z|=K} |p(z)| \int_r^R \frac{1}{K+t} dt \\ &\leq M(p,r) + \frac{nM(p,r)}{(1+r)^n} \int_r^R \frac{(K+t)^n}{K+t} dt - n \min_{|z|=K} |p(z)| \int_r^R \frac{1}{K+t} dt \\ &= M(p,r) + \frac{nM(p,r)}{(1+r)^n} \left[(K+R)^n - (K+r)^n \right] \frac{1}{n} - n \min_{|z|=K} |p(z)| \ln \left(\frac{K+R}{K+r} \right). \end{aligned}$$

Therefore, we get

$$M(p,r) \left[\frac{(1+r)^n + (K+R)^n - (K+r)^n}{(1+r)^n} \right] \geq M(p,R) + n \min_{|z|=K} |p(z)| \ln \left(\frac{K+R}{K+r} \right)$$

which is equivalent to

$$M(p,r) \geq \frac{(1+r)^n}{(1+r)^n + (K+R)^n - (K+r)^n} \left[M(p,R) + n \min_{|z|=K} |p(z)| \ln \left(\frac{K+R}{K+r} \right) \right],$$

and the proof of the theorem is now complete. \square

5 Examples

Although, in general, for any polynomial having no zeros on $|z| = 1$, our Theorem 2.3 always gives a bound sharper than obtainable by the known results, however in this section we present an example of a polynomial to show that in some cases the improvement can be considerably significant, and we do this by using MATLAB.

Example

- (a). Let $p(z) = z^3 + 64$, a polynomial of degree $n=3$. Then by using MATLAB, one can see that the zeros of this polynomial are : -4 , $2+3.4641i$, and $2-3.4641i$, hence $p(z) \neq 0$ in $|z| < 1$. If we use Theorem 1.3 with $R=0.5$ and $r=0.1$, we get

$$M(p,r) \geq (0.3943704)M(p,R).$$

Note that, for this polynomial $m=63$, so on using Corollary 2.1 of our Theorem 2.3, we easily get

$$M(p,r) \geq (0.3943704)M(p,R) + 23.117715,$$

an improvement of more than 23 over the bound obtained by Theorem 1.3.

- (b). If in the above example we take $R=1$ and $r=0.1$, as in (a), then we can apply Rivlin's Theorem 1.2, and get

$$M(p,r) \geq (0.166375)M(p,R),$$

while Corollary 2.3 of our Theorem 2.3 gives

$$M(p,r) \geq (0.166375)M(p,R) + 18.79891,$$

which is an improvement of about 18.8 over the bound obtained by Rivlin's Theorem 1.2.

Remark. It may be remarked that in fact one can always construct a polynomial for which this improvement is greater than any given positive number.

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